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A LIPSCHITZ DECOMPOSITION OF MINIMAL SURFACES

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1. Introduction

Let Γ be a simple closed rectifiable curve in Euclidean space $\mathbf{R}^n$. We say that Γ is an M *chord-arc curve* if $l(z, w) \leq M|z-w|$ for all $z, w \in \Gamma$, where $l(z, w)$ denotes the length of the shorter subarc of Γ joining z to w. Let $\psi(e^{it})$, $0 \leq t \leq 2\pi$, parametrize such a curve Γ with $|\psi'(e^{it})| \equiv l(\Gamma)/2\pi$, where $l(\Gamma)$ denotes the length of Γ. Then for $0 \leq t-s \leq \pi$, we have

$$c_1 \leq \frac{|\psi(e^{it}) - \psi(e^{is})|}{|e^{it} - e^{is}|} \leq c_2 \tag{1.1}$$

with $c_2/c_1 \leq \frac{\pi}{2}M$. In other words, Γ is a bi-Lipschitz image of the unit circle. Conversely, if (1.1) holds for some parametrization of Γ, then

$$l(\psi(e^{it}), \psi(e^{is})) \leq \left(\frac{\pi}{2}\right)^2 M|\psi(e^{it}) - \psi(e^{is})| \tag{1.2}$$

and thus Γ is a $(\frac{\pi}{2})^2 M$ chord-arc curve.

By a *minimal surface with boundary* Γ we mean the image $F(\mathbb{D})$ of the open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ under a continuous map

$$F = (F_1, \cdots, F_n): \overline{\mathbb{D}} \to \mathbb{R}^n$$

from the closed disk to $\mathbb{R}^n$ such that

(1.3) $F|_{\partial\mathbb{D}}$ is a homeomorphism of ∂D onto Γ,

(1.4) $F|_{\mathbb{D}}$ is C^2,

(1.5) $f_j \equiv \dfrac{\partial F_j}{\partial x} - i\dfrac{\partial F_j}{\partial y}$, $\quad 1 \leq j \leq n$, $z = x + iy$, is analytic in $\mathbb{D}$,

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and

$$\sum_{j=1}^{n} f_j^2(z) \equiv 0 \quad \text{in } \mathbb{D}. \tag{1.6}$$

Condition (1.5) says that each component F_j of F is a harmonic function in $\mathbb{D}$, and (1.6) says that the map F is angle preserving except at the (isolated) common zeros of $\{f_j\}$. By a famous theorem of Douglas [1] every simple closed curve Γ bounds at least one such minimal surface. We refer to Osserman's beautiful book [4] for further background on minimal surfaces.

By a *partition* of a domain $\Omega \subset \mathbb{D}$, we mean a family $\{D_j\}$ of simply connected subdomains of Ω such that

$$D_j \cap D_k = \varnothing, \qquad \text{if } j \neq k, \tag{1.7}$$

and

$$\Omega = \bigcup_j (\Omega \cap \overline{D}_j). \tag{1.8}$$

We will call such a partition *locally finite* if each compact subset of $\mathbb{D}$ meets at most a finite number of D_j. In this paper, we prove the following:

Theorem. *There is a universal constant M such that if Γ is a rectifiable simple closed curve in $\mathbb{R}^n$ and $F(\mathbb{D})$ is a minimal surface with boundary Γ, then there is a locally finite partition $\{D_j\}$ of $\mathbb{D}$ such that*

(1.9) *F is a homeomorphism of $\overline{D}_j$ onto $\overline{F(D_j)}$,*

(1.10) *$F(\partial(D_j))$ is an M chord-arc curve,*

and

$$\sum lF(\partial(D_j)) \leq Ml(\Gamma), \tag{1.11}$$

where $l(E)$ denotes the linear measure (or arc length) of the set E.

The only hard part of the theorem is inequality (1.11). Otherwise we could simply take each D_j to be a small square. When $n = 2$, $F_1 + iF_2$ is a conformal map to a plane domain with rectifiable boundary, and then the theorem is a recent result of Jones [3]. Our proof is a refinement of the argument from [3], where the estimate $(1-|z|^2)|f'|/|f| \leq 6$ is used in an essential way. When $n > 2$, the gradient $f = (f_1, \cdots, f_n)$ can have zeros in $\mathbb{D}$, and the example $f(z) = (1, -i, Nz, -iNz)$ shows that the above estimate can fail even if f does not have zeros. In the proof we will obtain curves that are actually better than M chord-arc. They can be taken to be arbitrarily close to planar M-Lipschitz curves, as defined in [3]. This improvement will be described in §5. We write

$$|f| = \left(\sum_{j=1}^{n} |f_j|^2\right)^{1/2}$$

and

$$f' = (f_1', \cdots, f_n').$$

Throughout the paper c, c_1, C, etc. stand for universal undetermined constants.

2. Preliminaries

The proof of (1.11) rests ultimately on the next lemma, an F. and M. Riesz theorem for minimal surfaces. The *Hardy space* H^1 is the set of g, analytic on $\mathbb{D}$, with

$$\|g\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |g(re^{i\theta})|\, d\theta < \infty.$$

Lemma 2.1. *If $F(\mathbb{D})$ is a minimal surface with rectifiable boundary Γ, then $f_j = \partial F_j/\partial x - i\partial F_j/\partial y \in H^1$, $1 \le j \le n$, and*

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta = \sqrt{2}\, l(\Gamma). \tag{2.1}$$

Proof. By (1.5) each F_j is the Poisson integral of its boundary values, and since Γ is rectifiable, each $F_j(e^{i\theta})$ is of bounded variation. Hence there are finite signed measures μ_j on $\partial\mathbb{D}$ so that $d\mu_j = (\partial F_j(e^{i\theta})/\partial\theta)\, d\theta$ and the vector measure $\mu = (\mu_1, \cdots, \mu_n)$ satisfies

$$\|\mu\| = \sup\left\{\sum_{j=1}^{n} \int h_j\, d\mu_j : h_j \text{ is continuous and } \sum h_j^2 \le 1\right\} = l(\Gamma).$$

Then $\partial F_j(z)/\partial\theta$, where $z = re^{i\theta}$, is the Poisson integral of μ_j, so that

$$\sup_{0<r<1} \int_0^{2\pi} \left\{\sum_{j=1}^{n} \left(\frac{\partial F_j(z)}{\partial\theta}\right)^2\right\}^{1/2} d\theta = l(\Gamma).$$

But by (1.5) and (1.6),

$$\sum_{j=1}^{n} \left(\frac{\partial F_j(z)}{\partial\theta}\right)^2 = \sum_{j=1}^{n} r^2 \left(\frac{\partial F_j}{\partial x}\right)^2 = r^2 \frac{|f|^2}{2},$$

and so (2.1) holds. q.e.d.

As an aside, we note this consequence of the lemma: If $F(\mathbb{D})$ is a minimal surface with rectifiable boundary Γ, and if G_j is analytic with $G_j' = f_j$ and $G_j(0) = F_j(0)$, then $F_j = \operatorname{Re} G_j$ and by the lemma G_j is continuous on $\overline{\mathbb{D}}$ and has bounded variation on $\partial\mathbb{D}$. Hence $G = (G_1, \cdots, G_n)$ is an analytic map of $\mathbb{D}$ into $\mathbb{C}^n$, G is a homeomorphism of $\partial\mathbb{D}$ onto the rectifiable curve $G(\partial\mathbb{D})$, and

$$l(G(\partial\mathbb{D})) = \sqrt{2}\, l(\Gamma). \tag{2.2}$$

Therefore $F(\mathbb{D})$ is the projection onto $\mathbb{R}^n$ of the analytic variety $G(\mathbb{D})$ in $\mathbb{C}^n = \mathbb{R}^{2n}$ for which (2.2) holds.

A measure σ on $\overline{\mathbb{D}}$ is a *Carleson measure* if there is a constant B such that for all θ_0 and all s, $0 < s \le 1$,

$$\sigma(\{re^{i\theta} : 1 - s \le r \le 1,\ \theta_0 \le \theta \le \theta_0 + s\}) \le Bs. \tag{2.3}$$

The Carleson norm $\|\sigma\|$ of σ is the least such B. By Carleson's theorem (see p. 62 of [2]), there is a constant A (independent of σ) so that (2.3) implies

$$\int_{\overline{\mathbb{D}}} |g|\, d\sigma \le A\|\sigma\|\, \|g\|_{H^1}$$

for all $g \in H^1$.

Our strategy will be to partition $\mathbb{D}$ into regions D_j so small that f is almost constant on D_j, yet so large that arc length on $\bigcup \partial D_j$ is a Carleson measure. Constructions of this type are well known; they stem from Carleson's proof of the corona theorem and are based on the following decomposition of $\mathbb{D}$.

For $m \ge 1$ and $1 \le j \le 2^{m+1}$, form the *dyadic squares*

$$Q_{m,j} = \{re^{i\theta} : (j-1)2^{-m}\pi \le \theta < j2^{-m}\pi;\ 1 - \pi 2^{-m} \le r < 1\}$$

(when $m = 1$, we require $r \ge 0$), and their *top halves*

$$T(Q_{m,j}) = Q_{m,j} \setminus \bigcup_k Q_{m+1,k}.$$

Fix an integer $N \ge 1$ and refine the dyadic grid by defining *small squares*

$$\begin{aligned} S = S_{m,j,p,q} &= \{re^{i\theta} : 2^{-m}\pi[(j-1) + (q-1)2^{-N}] \le \theta < 2^{-m}\pi[(j-1) + q2^{-N}]; \\ &\qquad 1 - 2^{-m}\pi[\tfrac{1}{2} + p2^{-N}] \le r < 1 - 2^{-m}\pi[\tfrac{1}{2} + (p-1)2^{-N}]\}, \end{aligned}$$

where m, j, p, and q are integers with $m \ge 1$, $1 \le j \le 2^{m+1}$, $1 \le q \le 2^N$, and $1 \le p \le 2^{N-1}$. In other words, each $T(Q_{m,j})$ is to be divided into $4^N/2$ small squares S with edge length $l(\partial S)$ approximately $4\pi 2^{-m-N}$. When E is any subset of $\mathbb{D}$, let $E^* = \{e^{i\theta}: re^{i\theta} \in E$ for some $r \geq 0\}$ denote its projection on $\partial\mathbb{D}$. For S a small square define $Q(S) = \{re^{i\theta}: e^{i\theta} \in S^*,\ 1 - \pi 2^{-m-N} \leq r < 1\}$ as the dyadic square having $Q(S)^* = S^*$, and define $B(S) = \{re^{i\theta}: e^{i\theta} \in S^*; \inf_{z\in S}|z| \leq r < 1 - \pi 2^{-m-N}\}$ as the *tower* which includes S but not $Q(S)$. Note that the aspect ratio $l(\partial B(S))/l(S^*)$ is essentially constant, once N is fixed. A region of the form

$$\mathscr{D} = Q \setminus \bigcup_{S\in\mathscr{S}(Q)} \overline{B(S)} \cup \overline{Q(S)}, \tag{2.5}$$

where $\mathscr{S}(Q)$ is some subcollection of small squares, has boundary an M_0 chord-arc curve, where M_0 depends on N but not on the subcollection $\mathscr{S}(Q)$. This is because each maximal $B(S) \cup Q(S)$ not in $\mathscr{D}$ is either adjacent to a larger tower not in $\mathscr{D}$ or at a distance at least $l(S^*)$ from any larger tower not in $\mathscr{D}$. Moreover, such regions $\mathscr{D}$ satisfy

$$l(\partial D \cap Q') \leq K l(\partial Q')$$

for every dyadic square Q', where K is a constant depending only on N. Thus, by Carleson's theorem,

$$\int_{\partial\mathscr{D}} |g|\, ds \leq AK \int_{\partial\mathbb{D}} |g|\, d\theta \tag{2.6}$$

for all $g \in H^1$, where ds is arc length measure.

3. Chord-arc curves

In this section we give three ways to obtain chord-arc curves in $\mathbb{R}^n$.

Lemma 3.1. *Suppose that* γ *is an* M *chord-arc curve in* $\mathbb{D}$, *and that there is a* $z_0 \in \mathbb{D}$ *with* $|f(z) - f(z_0)| < \delta |f(z_0)|$ *for all* $z \in \gamma$, *where* $\delta < 1/(\sqrt{2}M)$. *Then* $F(\gamma)$ *is an* M_1 *chord-arc curve, where* $M_1 = (\pi/2)^3((1 + \sqrt{2}\delta M)/(1 - \sqrt{2}\delta M))M$.

Proof. Suppose $\psi(e^{it})$ is a parametrization of γ with $|\psi'(e^{it})| = l(\gamma)/(2\pi)$ for all t. Fix s and t. By a rotation we may suppose $\psi(e^{it}) - \psi(e^{is}) \in \mathbb{R}$. By (1.5), (1.6), and the definition of M chord-arc curve,

$$\begin{aligned}
&|F(\psi(e^{it})) - F(\psi(e^{is})) - \operatorname{Re}(f(z_0))(\psi(e^{it}) - \psi(e^{is}))| \\
&\quad = \left| \int_s^t \operatorname{Re}[(f(\psi(e^{iu})) - f(z_0))\psi'(e^{iu})ie^{iu}]\,du \right| \\
&\quad \leq \delta |f(z_0)| l(\psi(e^{it}), \psi(e^{is})) \\
&\quad \leq \delta\sqrt{2} |\operatorname{Re} f(z_0)| M |\psi(e^{it}) - \psi(e^{is})|.
\end{aligned}$$

We conclude

$$|\operatorname{Re} f(z_0)|(1-\sqrt{2}\delta M) \leq \frac{|F(\psi(e^{it})) - F(\psi(e^{is}))|}{|\psi(e^{it}) - \psi(e^{is})|} \leq |\operatorname{Re} f(z_0)|(1+\sqrt{2}\delta M).$$

By (1.1) and (1.2), $F(\gamma)$ is an M_1 chord-arc curve with

$$M_1 \leq \left(\frac{\pi}{2}\right)^2 \frac{1+\sqrt{2}\delta M}{1-\sqrt{2}\delta M} M. \qquad \text{q.e.d.}$$

Near a zero of f, we cannot have an inequality like that required in Lemma 3.1. If $f(0) = 0$, write

$$f(z) = az^m + O(z^{m+1}), \tag{3.1}$$

where $a \in \mathbb{C}^n$, $a \neq 0$. Let $D_r = \{z: |z| < r\}$ and $D_{j,r} = \{se^{i\theta} \in D_r: (j-1)\pi/(m+1) \leq \theta < j\pi/(m+1)\}$, for $j = 1, \cdots, 2(m+1)$.

Lemma 3.2. *Suppose f has the form (3.1). If r is sufficiently small, then $F(\partial D_{j,r})$ is an M chord-arc curve with M independent of a and m and*

$$\sum_{j=1}^{2(m+1)} l(F(\partial D_{j,r})) \leq 2l(F(\partial D_r)).$$

Proof. Let $\psi(z) = z^{1/(m+1)}$ and consider $G \equiv F \circ \psi$ on the boundary of the half disk $D^+ = \{z: |z| < r^{m+1}, \operatorname{Im} z > 0\}$. Then $g \equiv (f \circ \psi)\psi' = a/(m+1) + O(z^{1/(m+1)})$. So if r is sufficiently small, then

$$\left| g(z) - \frac{a}{m+1} \right| < \delta \left| \frac{a}{m+1} \right|.$$

Since the boundary of a half disk is an M_1 chord-arc curve by Lemma 3.1, $F(\partial D_{1,r})$ is a $4M_1$ chord-arc curve if δ is sufficiently small. By rotating ψ, the same is true for $F(\partial D_{j,r})$, $2 \leq j \leq 2(m = 1)$. Moreover

$$l(F(\partial D_{j,r})) = \int_{\partial D^+} |g|\,ds \leq 2\int_{\partial D^+ \cap \{\operatorname{Im} z>0\}} |g|\,ds = 2l(F(\partial D_{j,r} \cap \partial D_r)).$$

Summing over j completes the proof. q.e.d.

The third method of constructing M chord-arc curves follows the argument given in [3, §2].

Lemma 3.3. *Given $\eta > 0$ there is a constant M depending only on η, so that if $\eta \le |f| \le 1$ on a simply connected domain $\mathscr{D} \subset \mathbb{D}$, then there is a partition $\{\mathscr{D}_j\}$ of $\mathscr{D}$ such that*

(3.2) $$\text{each } F(\partial \mathscr{D}_j) \text{ is an } M \text{ chord-arc curve}$$

and

(3.3) $$\sum l(F(\partial \mathscr{D}_j)) \le M l(F(\partial D)).$$

Moreover, if each component of $\partial\mathscr{D} \cap \mathbb{D}$ is smooth, then the partition $\{\mathscr{D}_j\}$ can be taken to be locally finite.

Proof. Let $G = F \circ \psi$ and $g = (f \circ \psi)\psi'$, where ψ is a conformal map of $\mathbb{D}$ onto $\mathscr{D}$. By Green's theorem,

$$\int_{\mathbb{D}} \Delta(|g|) \log \frac{1}{|z|} \frac{dx\,dy}{2\pi} = \int_{\partial\mathbb{D}} |g(e^{i\theta})| \frac{d\theta}{2\pi} - |g(0)|,$$

and by the Cauchy-Schwarz inequality,

$$\Delta(|g|) = \frac{2|g'|^2}{|g|} - \frac{|\langle g', g\rangle|^2}{|g|^3} \ge \frac{|g'|^2}{|g|}.$$

Hence we obtain the inequalities

$$\int_{\mathbb{D}} \frac{|g'|^2}{|g|} \log \frac{1}{|z|} \frac{dx\,dy}{2\pi} \le \int_{\partial\mathbb{D}} |g(e^{i\theta}| \frac{d\theta}{2\pi} - |g(0)| \le 2 \int_{\mathbb{D}} \frac{|g'|^2}{|g|} \log \frac{1}{|z|} \frac{dx\,dy}{2\pi}.$$

We also need the estimate

$$\frac{|g'|}{|g|} \le \frac{|(f \circ \psi)'|}{|f \circ \psi|} + \frac{|\psi''|}{|\psi'|} \le \frac{K}{1 - |z|^2},$$

where K is a constant depending only on η; it follows because $\log \psi'$ is in the Bloch space with Bloch norm independent of ψ and because $\eta \le |f \circ \psi| \le 1$.

We now repeat the stopping time argument of §2 of [3], slightly modified to ensure that our partition of $\mathbb{D}$ is locally finite. For a dyadic square Q, we define a subregion $\mathscr{D}_Q$ as follows: If there is a $z \in T(Q)$ with $|g(z) - g(z_Q)| \ge \frac{\delta}{2}|g(z_Q)|$, where z_Q is the center of $T(Q)$, stop and let $\mathscr{D}_Q = T(Q)$. In this case we say $\mathscr{D}_Q$ is of type 0. Otherwise, let $\{Q_j\}$ be those dyadic squares inside Q, which satisfy

$$\sup_{z \in T(Q_j)} |g(z) - g(z_Q)| \ge \delta |g(z_Q)|$$

and define $\mathscr{D}_Q = Q \setminus \bigcup_{j=1}^{\infty} \overline{Q}_j$. We say such a $\mathscr{D}_Q$ is of type 1 if $l(\partial\mathbb{D} \cap \partial\mathscr{D}_Q) \geq \frac{1}{2} l(\partial\mathbb{D} \cap \partial Q)$, and we say $\mathscr{D}_Q$ is of type 2 otherwise. The reason for using $\frac{\delta}{2}$ is that if $\zeta \in \partial\mathbb{D}$ and $\psi(\zeta) \in \mathbb{D}$, then g is continuous and nonzero at ζ, so the stopping time argument near ζ will eventually yield a dyadic square Q on which $|g(z) - g(z_Q)| < \delta |g(z_Q)|$, i.e., $\mathscr{D}_Q = Q$.

Since each component of g may have a zero in $\mathbb{D}$, we avoid the use of $g^{1/2}$ used to prove (2.8) of [3] by the following slight modification of the argument therein: As in [3], there is a δ' depending on δ and K such that for type 2 regions

$$\delta' |g(z_Q)|^2 \leq \int_{\partial\mathscr{D}_Q} |g - g(z_Q)|^2 \, d\omega = \int_{\mathscr{D}_Q} |g'|^2 \mathscr{G}_{z_Q}(z) \, dx \, dy,$$

where $\mathscr{G}_{z_Q}$ is Green's function in $\mathscr{D}_Q$ with pole at z_Q, and $d\omega = \frac{\partial \mathscr{G}}{\partial \eta} \frac{|dz|}{2\pi}$ is harmonic measure on $\partial\mathscr{D}_Q$ for the point z_Q. As in [3], the latter quantity is at most

$$\frac{C}{l(Q^*)} \int_{\mathscr{D}_Q} |g'(z)|^2 \log \frac{1}{|z|} \, dx \, dy \leq C_1 \frac{|g(z_Q)|}{l(\partial\mathscr{D}_Q)} \int_{\mathscr{D}_Q} \frac{|g'|^2}{|g|} \log \frac{1}{|z|} \, dx \, dy.$$

Hence

$$\int_{\partial\mathscr{D}_Q} |g| \, |dz| \leq (1 + \delta) \frac{|g(z_Q)|^2 l(\partial\mathscr{D}_Q)}{|g(z_Q)|} \leq K_1 \int_{\mathscr{D}_Q} |g'|^2 |g| \log \frac{1}{|z|} \, dx \, dy,$$

where K_1 is a constant depending on δ'.

The stopping time argument (so modified) in §2 of [3] can now be repeated to yield a partition $\widetilde{\mathscr{D}}_j$ of $\mathbb{D}$ such that each $\widetilde{\mathscr{D}}_j$ is an M_1 chord-arc curve, $|g(z) - a_j| < \delta |a_j|$ on $\widetilde{\mathscr{D}}_j$ for some $a_j \in C^n$, and, by letting $\mathscr{D}_j = \psi(\widetilde{\mathscr{D}}_j)$,

$$\sum_j l(F(\partial\mathscr{D}_j)) = \sum_j l(G(\partial\widetilde{\mathscr{D}}_j)) \leq M_1 l(G(\partial\mathbb{D})) = M1 l(F(\partial\mathscr{D})).$$

By Lemma 3.1, each $F(\mathscr{D}_j) = G(\widetilde{\mathscr{D}}_j)$ is an M chord-arc curve.

4. When f is small

In this section, we remove the hypothesis that $|f| \geq \eta > 0$ in Lemma 3.3.

Lemma 4.1. *There is a constant M so that if $|f| \leq 1$ on a simply connected domain $\mathscr{D} \subset \mathbb{D}$, then there is a partition $\{\mathscr{D}_j\}$ of $\mathscr{D}$ such that*

(4.1) *each* $F(\partial\mathscr{D}_j)$ *is an* M *chord-arc curve*

and

(4.2) $$\sum l(F(\partial\mathscr{D}_j)) \le Ml(\partial\mathscr{D}).$$

Moreover, if each component of $\partial\mathscr{D}\cap\mathbb{D}$ *is smooth, then the partition* $\{\mathscr{D}_j\}$ *can be taken to be locally finite.*

Notice that (4.2) is a weaker conclusion than (3.3).

Proof. Our strategy will be to divide $\mathscr{D}$ into good regions and bad regions. Lemma 3.3 will apply to the good regions, and f will be small on the bad regions. The process will be restarted on each bad region $\mathscr{B}$ with f replaced by $f/\sup_{\mathscr{B}}|f|$. Let φ be a conformal map of $\mathbb{D}$ onto $\mathscr{D}$. We will subdivide certain dyadic squares Q into two cases:

Fix $\alpha > 0$, $\varepsilon > 0$, and an integer N, where α, ε and N' are to be chosen later, with $\varepsilon < \alpha/2$.

Case 1: $\sup_{T(Q)}|f\circ\varphi| \le \alpha/2$. Define descendent squares Q_j to be the maximal dyadic squares contained in Q, for which

$$\sup_{T(Q_j)}|f\circ\varphi| \ge \alpha,$$

and let $\mathscr{B} = \mathscr{B}(Q) = Q\setminus\cup\overline{Q}_j$ be called a *bad region of the first kind.* Note that $|f\circ\varphi| \le \alpha$ for all $z\in\mathscr{B}$.

Case 2: $|f\circ\varphi| > \alpha/2$. Let $\mathscr{S}(Q)$ be the set of small squares $S\subset Q$ such that

$$\inf_S |f\circ\varphi| \le \varepsilon$$

and such that its projection S^* and its tower $B(S)$ are maximal. The descendent squares $\{Q_j\}$ are defined to be $\{Q(S) : S\in\mathscr{S}(Q)\}$. Each component $\mathscr{G}_j$ of $Q\setminus\bigcup\{\overline{B(S)}\cup\overline{Q(S)} : S\in\mathscr{S}(Q)\}$ is declared a *good region of the first kind.* Inside the towers $B(S)$, $S\in\mathscr{S}(Q)$, we must define other good and bad regions. By a *very small square* we mean a square of the form given in (2.4) with N replaced by $N+N'$. So if S' is a very small square contained in a small square S, then $l(\partial S')$ is approximately $2^{-N'}l(\partial S)$. There are $4^{N'}$ such very small squares S' in each small square S. Let $\mathscr{S}'(S)$ be the set of very small squares $S'\subset B(S)$ that either contain a zero of $f\circ\varphi$ or touch a very small square containing a zero of $f\circ\varphi$. In other words, $\mathscr{S}'(S) = \{S' : S'\subset B(S)$ and $\overline{S'}\cap\overline{S''}\ne\varnothing$ for some S'' containing a zero of $f\circ\varphi$, where S' and S'' are very small squares$\}$. Each $S'\in\mathscr{S}'(S)$ will be declared a *bad region of* *the second kind*, and each $S' \notin \mathcal{S}'(S)$ with $S' \subset B(S)$ will be declared a *good region of the second kind.* If N' is sufficiently large, by Schwarz's lemma $|f \circ \varphi| < \alpha$ on each $S' \in \mathcal{S}'(S)$, since S' is near a zero of $f \circ \varphi$. Thus $|f \circ \varphi| \le \alpha$ on all bad regions.

In order to apply Lemma 3.3 to each good region, we need to see that $|f \circ \varphi|$ is not too small there. On each good region of the first kind $|f \circ \varphi| \ge \varepsilon$ by construction. To obtain a similar estimate for good regions of the second kind, we first estimate the number of zeros of $f \circ \varphi$ near a tower $B(S)$, $S \in \mathcal{S}(Q)$. Suppose $\inf_{z \in S} |z| > \inf_{z \in Q} |z|$. Then $|f \circ \varphi| \ge \varepsilon$ on the top edge, $\{\zeta \in \overline{S} : |\zeta| = \inf_{z \in S} |z|\}$ of $B(S)$, and hence there is a unit vector $u = (u_1, \cdots, u_n)$ so that the function $g = f \circ \varphi \cdot u$ satisfies $|g(\zeta)| \ge \varepsilon$ for some ζ on the top edge of $B(S)$. Let $\widetilde{B}(S) = \bigcup \{\widetilde{S} : \widetilde{S}$ is a small square with $\operatorname{dist}(\widetilde{S}, B(S)) < l(\partial S)/8\}$ and let $Z(S) = \{z_v \in \widetilde{B}(S) : g(z_v) = 0\}$. By p. 288 of [2] again,

$$\sum_{z_v \in Z(S)} \operatorname{Im} z_v \le C_3 2^N l(\partial S) \log 1/\varepsilon . \tag{4.3}$$

Since $\operatorname{Im} z_v \ge l(\partial S)/16$, we see that there are at most $K(\varepsilon, N) = 1 + C_4 2^N \log 1/\varepsilon$ points in $Z(S)$, counting multiplicity. Since $|g| \ge \varepsilon$ at some point on the top edge of $B(S)$, Harnack's inequality shows that if z belongs to a good region $S' \subset B(S)$, then

$$|g(z)| \ge k(N) \delta^{K(\varepsilon, N)} \equiv \eta > 0 , \tag{4.4}$$

where $k(N)$ is a constant depending only on N, and $\delta = 2^{-N} 2^{-N'}$ is a lower bound for the pseudohyperbolic size of a "very small" square. If $\inf_{z \in S} |z| = \inf_{z \in Q} |z|$, inequality (4.4) persists since $l(\partial S)$ and $l(\partial Q)$ are comparable and $\sup_{T(Q)} |g| \ge \alpha/2 > \varepsilon$ for an appropriate unit vector u. Thus we conclude that $\eta \le |f \circ \varphi| \le 1$ on good regions of either kind. This argument also shows that there are at most $C_5 K(\varepsilon, N)$ bad regions of the second kind in each $B(S)$.

We note that the bad regions can be slightly increased and the neighboring good regions decreased, so that no zero of $f \circ \varphi$ occurs on the boundary of a bad region, and we still have $|f \circ \varphi| < \alpha$ on each bad region.

We apply the processes described in Cases 1 and 2 as follows. Beginning with each $Q_{1,k}$, as defined in §2, apply the appropriate Case 1 or Case 2 obtaining (in particular) descendent squares Q_j. To each descendent square, apply the appropriate case, obtaining the next *generation* of descendents. Continue this process indefinitely.

We need the following proposition.

Proposition 4.2. *Given $\alpha > 0$, we can choose an integer N and an $\varepsilon_0 > 0$, so that for each Case 2 dyadic square Q, if $\varepsilon \le \varepsilon_0$ then*

$$\sum\{l(\partial Q_j) : Q_j \text{ is a descendent of } Q\} \le l(\partial Q)/100. \tag{4.5}$$

Proof. Since Q is a Case 2 square there is a unit vector $u = (u_1, \cdots, u_n)$ so that the function $g = f\circ\varphi\cdot u$ satisfies $\sup_{T(Q)}|g| > \alpha/2$. By Schwarz's lemma, we can choose N sufficiently large, depending on ε_0, so that if $\inf_S |g| \le \varepsilon$ then $\sup_S |g| \le 2\varepsilon_0$. Thus Theorem 3.2 on p. 334 of [2] shows we can choose an ε_0, depending on α, so that if $\varepsilon \le \varepsilon_0$

$$\sum\left\{l(S^*) : S \subset Q \text{ and } \inf_S |g| \le \varepsilon\right\} \le l(Q^*)/100,$$

which gives (4.5). q.e.d.

Since each descendent of a Case 1 square is a Case 2 square, this proposition yields that for any dyadic square Q', we have

$$\sum_{\mathscr{G}_i \text{ good}} l(\partial\mathscr{G}_i \cap Q') \le Kl(\partial Q'), \tag{4.6}$$

where K is a constant depending on N and N'. The proposition also implies that for N' sufficiently large,

$$\sum_{\mathscr{B}_i \text{ bad}} l(\partial\mathscr{B}_i \cap Q') \le C_6 l(\partial Q'), \tag{4.7}$$

where C_6 is a universal constant. To see this, note that if $\mathscr{B}$ is a bad region of the first kind, coming from a dyadic square Q, then $l(\partial\mathscr{B}) \le 2l(\partial Q)$. Furthermore, if S is a small square in $\mathscr{S}(Q)$, then our bound on the number of zeros near $B(S)$ gives

$$\begin{aligned}&\sum\{\partial\mathscr{B} : \mathscr{B} \text{ is a bad region of the second kind } \subset B(S)\}\\ &\quad\le C2^N(\log 1/\varepsilon)2^{-N'}l(\partial S) \le l(\partial S)\end{aligned}$$

for N' sufficiently large.

By Carleson's theorem, we obtain

$$\begin{aligned}\sum_{\mathscr{G}_i \text{ good}} \int_{\partial\varphi(\mathscr{G}_i)} |f|\,ds &= \sum_{\mathscr{G}_i \text{ good}} \int_{\partial\mathscr{G}_i} |f\circ\varphi|\,|\varphi'|\,ds\\ &\le CK\int_{\partial\mathbb{D}} |f\circ\varphi|\,|\varphi'|\,ds = CK\int_{\partial\mathscr{D}} |f|\,ds\end{aligned} \tag{4.8}$$

and

$$\sum_{\mathscr{B}_i \text{ bad}} \int_{\partial\mathscr{B}_i} |\varphi'|\,ds \le CC_6\int_{\partial\mathbb{D}} |\varphi'|\,ds = C_7 l(\partial\mathscr{D}). \tag{4.9}$$

We continue our subdivisions now at a second *level.* For each bad region $\mathscr{B}_i$, let ψ_i be a conformal map of $\mathbb{D}$ onto $\mathscr{B}_i$ and let $g = (f \circ \varphi \circ \psi_i)/\sup_{\mathscr{B}} |f \circ \varphi|$. If there is only one zero ζ of $f \circ \varphi$ in $\mathscr{B}_i$, we choose ψ_i so that $\psi_i(0) = \zeta$. In this case, we choose r so small that Lemma 3.2 applies to $(f \circ \varphi \circ \psi_i)(\varphi \circ \psi_i)'$ and $F \circ \varphi \circ \psi_i$. Since

$$\int_{|z|=r} |f \circ \varphi \circ \psi_i||\varphi' \circ \psi_i||\psi_i'|\,ds \le \int_{\partial\mathscr{B}} |f \circ \varphi|\,\varphi'|\,ds,$$

the small sectors from Lemma 3.2 will at most double the total length estimates. For notational convenience, we will call these sectors good regions. The initial regions $G_{1,j}$ are replaced in this case by $G_{1,j}\backslash\{z\colon |z| \le r\}$, $j = 1, \cdots, 4$.

Replacing φ with $\varphi \circ \psi_i$ and $f \circ \varphi$ with g, we apply the process described above to obtain a second level of good regions $\mathscr{G}_{i,j}^{(2)}$ and bad regions $\mathscr{B}_{i,j}^{(2)}$. Then by (4.6) and (4.9), we get

$$\begin{aligned}
&\sum_{\mathscr{B}_i^{(1)}\text{ bad}} \sum_{\mathscr{G}_{i,j}^{(2)}\text{ good}} \int_{\partial(\varphi\circ\psi(\mathscr{G}_{i,j}^{(2)}))} |f|\,ds \\
&\quad= \sum_{\mathscr{B}_i^{(1)}\text{ bad}} \sum_{\mathscr{G}_{i,j}^{(2)}\text{ good}} \int_{\partial\mathscr{G}_{i,j}^{(2)}} |f \circ \varphi \circ \psi_i|\,|(\varphi \circ \psi_i)'|\,ds \\
&\quad\le CK \sum_{\mathscr{B}_i^{(1)}\text{ bad}} \int_{\partial\mathbb{D}} |f \circ \varphi \circ \psi_i|\,|(\varphi \circ \psi_i)'\,ds \\
&\quad= CK \sum_{\mathscr{B}_i^{(1)}\text{ bad}} \int_{\partial\mathscr{B}_i^{(1)}} |f \circ \varphi|\,|\varphi'|\,ds \le CKC_7\alpha l(\partial\mathscr{D}).
\end{aligned}$$

Furthermore, a use of (4.7) and (4.9) yields

$$\begin{aligned}
\sum_{\mathscr{B}_i^{(1)}\text{ bad}} \sum_{\mathscr{B}_{i,j}^{(2)}\text{ bad}} \int_{\partial\mathscr{B}_{i,j}^{(2)}} |\varphi' \circ \psi_i|\,\psi_i'|\,ds &\le C_7 \sum_{\mathscr{B}_i^{(1)}\text{ bad}} \int_{\partial\mathbb{D}} |\varphi' \circ \psi_i|\,|\psi_i'|\,ds \\
= C_7 \sum_{\mathscr{B}_i^{(1)}\text{ bad}} \int_{\partial\mathscr{B}_i^{(1)}} |\varphi'|\,ds \le C_7^2 l(\partial\mathscr{D}).
\end{aligned}$$

For each bad region at the second level, we repeat this process obtaining third level good and bad regions. Continue this subdivision indefinitely. We obtain a partition of $\mathscr{D}$ into regions $\tau_k(\mathscr{G}_k)$, where each $\mathscr{G}_k$ is a good region at some level, and τ_k is a conformal map of $\mathbb{D}$ into $\mathscr{D}$. Indeed, $|f| \le \alpha^m$ on $\tau_k(\mathscr{B})$, where $\mathscr{B}$ is a bad region at level m, so each point of $\mathscr{D}\backslash\{z\colon f(z) = 0\}$ is in at most finitely many bad regions. Each zero of

f is eventually in a region $\tau_k(\mathscr{B})$ where Lemma 3.2 is applied to $f \circ \tau_k$ on $\mathscr{B}$. Choose α so that $C_7\alpha < 1$. Then

$$\begin{aligned}\sum_k \int_{\partial\tau_k(\mathscr{G}_k)} |f|\, ds &\leq CK[1 + C_7\alpha + (C_7\alpha)^2 + \cdots]l(\partial\mathscr{D}) \\ &= \frac{CK}{1 - C_7\alpha} l(\partial\mathscr{D}).\end{aligned} \tag{4.10}$$

In order to make our partition locally finite, we reduce the size of each good region $\mathscr{G}_k$ slightly, so that each component of $\tau_k(\mathscr{G}_k) \cap \mathbb{D}$ is smooth. Indeed, we can find almost square regions $\mathscr{D}_j' \subset \mathscr{G}_k$ so that

(i) for each j, there is an $a_j \in \mathbb{C}^m$ with $|f \circ \tau_k - a_j| < \delta |a_j|$ on $\mathscr{D}_j'$,

(ii) $\sum l(\partial\mathscr{D}_j') \leq 5l(\partial\mathscr{G}_k)$,

(iii) each $\partial\mathscr{D}_j'$ is a 5 chord-arc curve, and

(iv) $\mathscr{G}_k' = \mathscr{G}_k \setminus \bigcup \overline{\mathscr{D}_j'}$ has each component of $\{\zeta \in \partial\mathscr{G}_k' : \tau_k(\zeta) \in \mathbb{D}\}$ a smooth curve.

Moreover, since each component of $\{\zeta \in \partial\mathscr{G}_k : \tau_k(\zeta) \in \mathbb{D}\}$ consists of radial line segments and arcs of circles centered at the origin, the components $\mathscr{D}_j'$ can be chosen so small and so close to squares that

$$\int_{\partial\mathscr{D}_j'} |\tau_k'(z)|\,|dz| \leq 5 \int_{\partial\mathscr{D}_j' \cap \partial\mathscr{G}_k} |\tau_k'(z)|\,|dz|.$$

The $\{\mathscr{D}_j'\}$ look like a one-cell thick skin around (most of) $\partial\mathscr{G}_k$, with variable sized cells. Thus

$$\begin{aligned}\sum_j \int_{\partial\tau_k(\mathscr{D}_j')} |f|\, ds &\leq (1+\delta) \sum_j |a_j| \int_{\partial\mathscr{D}_j'} |\tau_k'|\, ds \\ &\leq \frac{5(1+\delta)}{1-\delta} \sum_j \int_{\partial\mathscr{D}_j' \cap \partial\mathscr{G}_k} |f \circ \tau_k|\,|\tau_k'|\, ds \\ &\leq 6 \int_{\partial\tau_k(\mathscr{G}_k)} |f|\, ds.\end{aligned} \tag{4.11}$$

We now apply Lemma 3.3 to each $\mathscr{G}_k'$. By (4.10) and (4.11) we have the desired partition of $\mathscr{D}$.

To see that the partition is locally finite when each component of $\partial\mathscr{D} \cap \mathbb{D}$ is smooth, first note that at each level the good regions have $\{\tau_k(\mathscr{G}_k)\}$ locally finite. This is because if $\zeta \in \partial\mathbb{D}$ and $\tau_k(\zeta) \in \mathbb{D}$, then $f \circ \tau_k$ is continuous at ζ, so our stopping time argument either ends with a bad region of the first kind containing a neighborhood of ζ in $\mathbb{E}$, i.e., when $|f(\tau_k(\zeta))| \leq \alpha/2$, or with a Case 2 good region of the first kind containing a neighborhood of ζ in $\mathbb{D}$, i.e., when $|f(\tau_k(\zeta))| > \alpha/2$. Each partition within a good region is locally finite by Lemma 3.3. Since $|f| \le \alpha^m$ at the mth level, each point $\zeta \in \mathbb{D}\backslash\{z: f(z) = 0\}$ is in at most finitely many $\tau(\mathscr{B})$, $\mathscr{B}$ a bad region, and each zero of f is eventually the only zero in $\tau(\mathscr{B})$, for some conformal map τ and bad region $\mathscr{B}$. For each zero ζ of f, then, the process terminates near ζ with the good regions generated by the application of Lemma 3.2. We conclude that our partition is locally finite.

5. When f is large

To remove the boundedness restriction on f, we apply the following decomposition. Choose $r_0 < 1$ so that if C_θ is the (open) convex hull of $e^{i\theta}$ and $\{z: |z| < r_0\}$, then $T(Q) \subset C_\theta$ whenever Q is a dyadic square with $e^{i\theta} \in Q^*$ (in fact, $r_0 = \sqrt{4/5}$ will work). Let

$$f^*(\theta) = \sup\{|f(z)|: z \in C_\theta\}.$$

Using the Hardy-Littlewood maximal theorem and Lemma 2.1, we obtain $\int_0^{2\pi} |f^*(\theta)|\,d\theta \le C\|f\|_{H^1} = C\sqrt{2}l(\Gamma)$. Now suppose that Q is a dyadic square with $2^{m-1} \le \sup_{T(Q)} |f| < 2^m$, where m is an integer. Define descendent squares $Q_k \subset Q$ to be the maximal dyadic squares contained in Q for which $\sup_{T(Q_k)} |f| \ge 2^m$. Let $\mathscr{D}^m = Q\backslash \bigcup \overline{Q_k}$. Note that for $e^{i\theta} \in Q^*$, $|f^*(\theta)| > 2^{m-1}$, $l(\partial\mathscr{D}^m) \le 6l(Q^*)$, and $|f/2^m| < 1$ on $\mathscr{D}^m$. Begin with each $Q_{1,j}$ forming the associated regions $\mathscr{D}^m$. For each descendent Q_k, repeat the process by forming regions $\mathscr{D}^{m+1}$. Continuing the process indefinitely, we obtain a decomposition of $\mathbb{D}$ into regions of the form $\mathscr{D}^m = Q\backslash \bigcup \overline{Q_k}$, where $\sup_{\mathscr{D}} |f/2^m| < 1$. We may reduce the regions $\mathscr{D}$ at each stage slightly, as we did in the proof of (4.11), so that $\partial\mathscr{D}^m \cap \mathbb{D}$ is smooth. By Lemma 4.1 applied to $F/2^m$, we can partition each $\mathscr{D}^m$ into regions $\mathscr{D}_i^m$ with

$$\sum_i l(F(\partial\mathscr{D}_i^m)) \le M_1 2^m l(\partial\mathscr{D}^m).$$

Regions $\mathscr{D}^{m'}$ formed from Q_k, where $\mathscr{D}^m = Q\backslash\overline{Q_k}$, have $m' > m$. Thus

$$\sum_{m,i} l(F(\partial\mathscr{D}_i^m)) \le M_2 \sum 2^m |\{\theta: f^*(\theta) > 2^{m-1}\}| \le M_2 C\|f\|_{H^1} = Ml(\Gamma)$$

and the theorem is proved in full generality.

Finally, we note that the regions $\{F(\mathscr{D}_i^m)\}$ of the above partition are better than M chord-arc. By the proofs of Lemmas 3.2 and 3.3, each such $\mathscr{D}$ is the image under some conformal map τ of a region Ω, bounded by an M chord-arc curve, with

$$|(f \circ \tau)\tau' - a| < \delta|a|, \qquad z \in \Omega, \tag{5.1}$$

for some $a \in \mathbb{C}^n$. The Ω's coming from Lemma 3.2 are half disks and the Ω's coming from §2 of [3] are called M-Lipschitz curves. Namely, each such Ω after a translation, notation, and dilation can be parametrized by $(r(\theta)\cos\theta, r(\theta)\sin\theta)$, $0 \le \theta \le 2\pi$, where $1/(1+M) \le r \le 1$ and $|r(\theta_1)-r(\theta_2)| \le M|\theta_1-\theta_1|$ for all θ_1 and θ_2. We define an *M-δ Lipschitz curve* in $\mathbb{R}^n$ to be a curve parametrized after a translation, rotation, and dilation by

$$(\gamma(e^{i\theta}) = (r(\theta)\cos\theta, r(\theta)\sin\theta, x_3(e^{i\theta}), \cdots, x_n(e^{i\theta})), \qquad 0 \le \theta \le 2\pi,$$

where $1/(1+M) \le r(\theta) \le 1$, $|r(\theta_1) - r(\theta_2)| \le M|\theta_1 - \theta_2|$ and $|x_j(e^{i\theta_1}) - x_j(e^{i\theta_2})| \le \delta|\theta_1 - \theta_2|$ for all $\delta_0 > 0$, we can arrange that $\delta \le \delta_0$ in (5.1). Thus by Lemma 3.1, given any $\delta > 0$, we can find an $M_1 < \infty$ so that $\mathbb{D}$ can be partitioned into regions D_j so that $\partial F(D_j)$ is an M_1-δ Lipschitz curve and (1.11) holds. These M_1-δ Lipschitz regions are images of M-Lipschitz regions Ω_j with the property that any two points in Ω_j can be connected by a path γ consisting of a radial line segment, followed by a circular line segment, followed by another radial segment, where the radial segments are no longer than their distance apart. By the proof of Lemma 3.1, $F \circ \tau_j$ must be one-to-one on $\overline{\Omega_j}$. These regions $F(D_j)$ thus look like small perturbations of planar M-Lipschitz curves that have been translated, rotated, and dilated in $\mathbb{R}^n$. This concludes the proof of the theorem.

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